Exercise 56

Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(t) = \frac{\sqrt{t}}{1+t^2}, \quad [0, 2]$$

Solution

Take the derivative of the function.

$$\begin{aligned}
f'(t) &= \frac{d}{dt}\left(\frac{\sqrt{t}}{1+t^2}\right) \\
&= \frac{\left[\frac{d}{dt}(\sqrt{t})\right](1+t^2) - \left[\frac{d}{dt}(1+t^2)\right]\sqrt{t}}{(1+t^2)^2} \\
&= \frac{\left(\frac{1}{2}t^{-1/2}\right)(1+t^2) - (2t)\sqrt{t}}{(1+t^2)^2} \\
&= \frac{\left(\frac{1}{2\sqrt{t}}\right)(1+t^2) - (2t)\sqrt{t}}{(1+t^2)^2} \times \frac{2\sqrt{t}}{2\sqrt{t}} \\
&= \frac{(1+t^2) - 4t^2}{2\sqrt{t}(1+t^2)^2} \\
&= \frac{1-3t^2}{2\sqrt{t}(1+t^2)^2}
\end{aligned}$$

Set what's in the numerator equal to zero, and set what's in the denominator equal to zero. Solve both equations for t.

$$1 - 3t^2 = 0 \qquad\qquad 2\sqrt{t}(1+t^2)^2 = 0$$

$$t^2 = \frac{1}{3} \qquad\qquad \sqrt{t} = 0 \quad \text{or} \quad 1 + t^2 = 0$$

$$t = -\frac{1}{\sqrt{3}} \quad \text{or} \quad t = \frac{1}{\sqrt{3}} \qquad\qquad t = 0 \quad \text{or} \quad t = \pm i$$

$t = 0$ and $t = 1/\sqrt{3}$ are within $[0, 2]$, so evaluate f at these values.

$$f(0) = \frac{\sqrt{0}}{1+0^2} = 0 \qquad \text{(absolute minimum)}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{\frac{1}{\sqrt{3}}}}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{3^{3/4}}{4} \approx 0.569877 \qquad \text{(absolute maximum)}$$

Now evaluate the function at the other endpoint of the interval.

$$f(2) = \frac{\sqrt{2}}{1+2^2} = \frac{\sqrt{2}}{5} \approx 0.282843$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $[0, 2]$. The graph of the function below illustrates these results.